

Eigenvalues Statistics for Restricted Trace Ensembles

by

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Abstract

Random matrices theory has been extensively studied since the early work of Wigner and Dyson. Random matrices are as effective mathematical reference models for the description of statistical properties in the spectra of complex physical systems. In this dissertation, the author, from a mathematical viewpoint, obtains certainly rigorous results for restricted trace random matrices (RTEs), which bear the same relationship to unconstrained random matrices ensembles as that of the microcanonical ensembles to the canonical ensembles in statistical physics [8].

The dissertation is divided into three parts. First, we briefly introduce three classical random matrices ensembles which are known as the Gaussian unitary ensemble (GUE), the Gaussian orthogonal ensemble (GOE) and the Gaussian symplectic ensemble (GSE). The corresponding probability density functions (p.d.f.'s.) of the eigenvalues are presented. Analogously, we review the Wishart (or Laguerre) random matrices ensembles. Based on such background, we can naturally obtain the restricted trace ensembles, and their corresponding joint p.d.f.s., especially the restricted trace Laguerre ensembles. After introducing the definition of the correlation functions of the eigenvalues for random matrices ensembles, we give a survey on some known results for the correlation functions, including the sine kernel, Airy kernel and Bessel kernel.

Secondly, we consider the fixed and bounded trace Gaussian orthogonal, unitary and symplectic ensembles. In case of the Gaussian ensembles, we prove the universal limits of correlation functions at zero and at the edge of the spectrum edge. The key idea is inspired by [43]. In addition, by using the universal result in the bulk for the fixed trace Gaussian unitary ensemble, which has been obtained by Götze and Gordin [42], the universal limits of correlation functions for the bounded trace Gaussian unitary ensembles are obtained.

Thirdly, we work on the fixed (bounded) trace unitary ensemble, naturally appears in quantum information and quantum chaos. We focus on correlation functions of Schmidt eigenvalues for the model and prove universal limits of the correlation functions in the bulk and also at the soft and hard edges of the spectrum, as that of the LUE. In the bulk of the spectrum, we make use of the asymptotic behavior of Laguerre polynomials in the complex plane, as developed in [97]. Further we consider the bounded trace LUE and obtain the same universal limits. On the other hand, for finite N , we compute the distribution of the smallest eigenvalue of fixed trace Laguerre ensemble. Furthermore, we found, based on the exact result obtained, a limit distribution, under a certain large N scaling. This turns out to

be the same as the smallest eigenvalue distribution of the classical Laguerre ensembles without the fixed trace constraint. This suggests that in a broad sense global constraint does not influence local correlations at least in the infinite matrix limit. Consequently, by tracing out the environmental degrees of freedom, we obtain the determination of the smallest eigenvalue distribution of the reduced density matrix for a bipartite quantum system of unequal dimensions.



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Declaration

The author declares that this thesis represents his own work with Professor Tao Qian, the author's supervisor and his collaborator Dang-Zheng Liu. The results in this thesis, unless otherwise stated or indicated, have not been previously included in any thesis, dissertation or report submitted to any institution for a degree, diploma or other qualification, or for publication by the author, and to the author's knowledge, by anyone else.



Da-Sheng Zhou

To
My Parents





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Abbreviations

RMT	Random matrix theory
GU(O,S)E	Gaussian Unitary(Orthogonal, Unitary) Ensemble
LU(O,S)Es	Laguerre Unitary(Orthogonal, Unitary) Ensembles
RTEs	Restricted trace ensembles
FTE	Fixed trace ensembles
BTE	bounded trace ensembles
FTLUE	Fixed trace Laguerre unitary ensemble
BTLUE	Bounded trace Laguerre unitary ensemble
RTLE(s)	Restricted trace Laguerre ensemble(s)
p.d.f.	probability density function
p.d.f's.	probability density functions



Basic symbols, notations

- We will use the notation $\vec{v}_n$ for the n -dimensional row vector $(v_1, \dots, v_n)$ without further explanation.
- The abbreviation $d\vec{v}_n$ denotes the Lebesgue measure $dv_1 \dots dv_n$ on $\mathbb{R}^n$.
- The symbol $\|v_n\|$ is expressed as the l^2 (Euclidean) norm of the vector $\vec{v}_n$.
- We will use the convention that the C 's denote generically bounded constants independent on N , whose values may depend on β and change from line to line.

