

**Norm inequalities for a matrix product analogous
to the commutator**

by

Io Kei Lok



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**Faculty of Science and Technology
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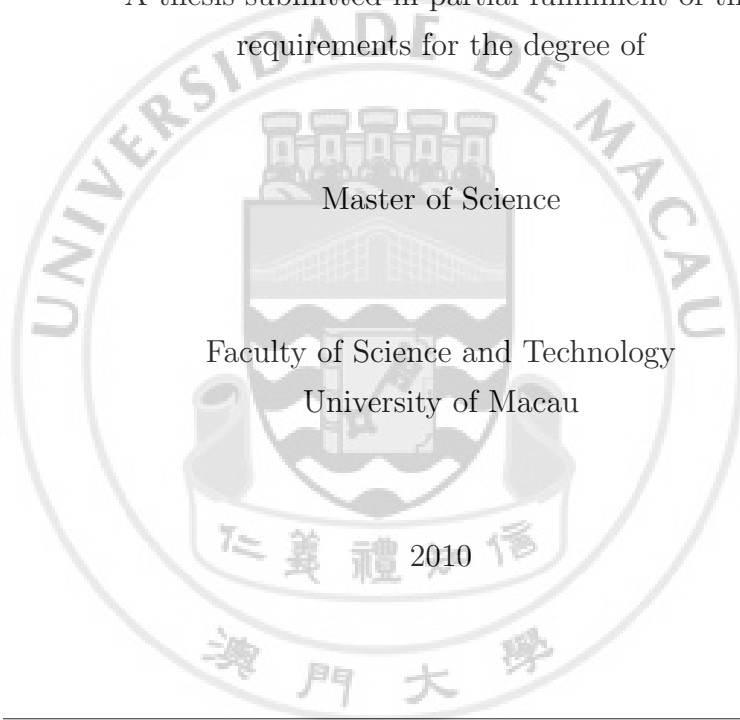
by

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A thesis submitted in partial fulfillment of the
requirements for the degree of

Master of Science

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University of Macau



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DECLARATION

The author declares that this thesis represents his own work with Professor Che Man Cheng and Dr. Ieng Tak Leong, the author's supervisor and co-supervisor. All the work is done under the supervision of Professor Cheng and Dr. Leong during the period 2008-2010 for the degree of Master of Science in Mathematics at the University of Macau. The results in this thesis, unless otherwise stated or indicated, have not been previously included in any thesis, dissertation or report submitted to any institution for a degree, diploma or other qualification, or for publication by the author, and to the author's knowledge, by anyone else.



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Abstract

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Thesis Supervisors:

Professor Che Man Cheng, Dr. Ieng Tak Leong

Master of Science in Mathematics

It has been conjectured and proved that

$$\|XY - YX\|_F \leq \sqrt{2}\|X\|_F\|Y\|_F,$$

for any $n \times n$ complex matrices X and Y , where $\|\cdot\|_F$ denotes the Frobenius norm. In this thesis, we show that the commutator $XY - YX$ in the above inequality can be replaced by the product $XY - YX^T$ for real matrices X and Y , where X^T denotes the transpose of X . The proof is given in Chapter 2. We also give the characterization of those pairs of matrices that satisfy the inequality with equality in Chapter 3.

Audenaert showed that for any $n \times n$ complex matrices X and Y , the above inequality can be strengthened as

$$\|XY - YX\|_F \leq \sqrt{2}\|X\|_F\|Y\|_{(2),2},$$

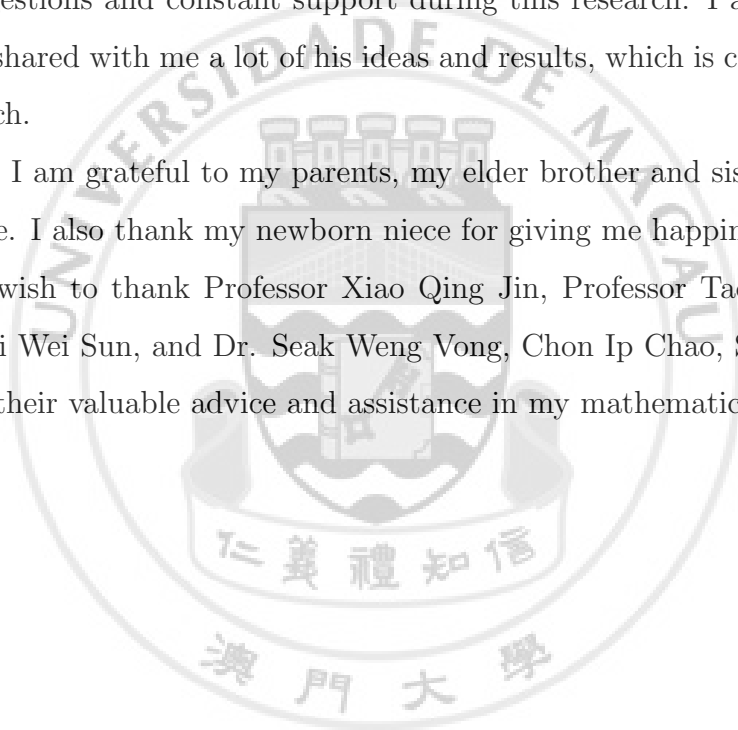
where $\|\cdot\|_{(2),2}$ denotes the $(2, 2)$ -norm. In Chapter 4 we show that the commutator $XY - YX$ in this inequality can also be replaced by the product $XY - YX^T$ for real matrices X and Y . Those pairs of matrices which satisfy the inequality with equality are also characterized.

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