

Norm Inequalities for Commutators

by

Kin Sio Fong



Master of Science

2010



**Faculty of Science and Technology
University of Macau**

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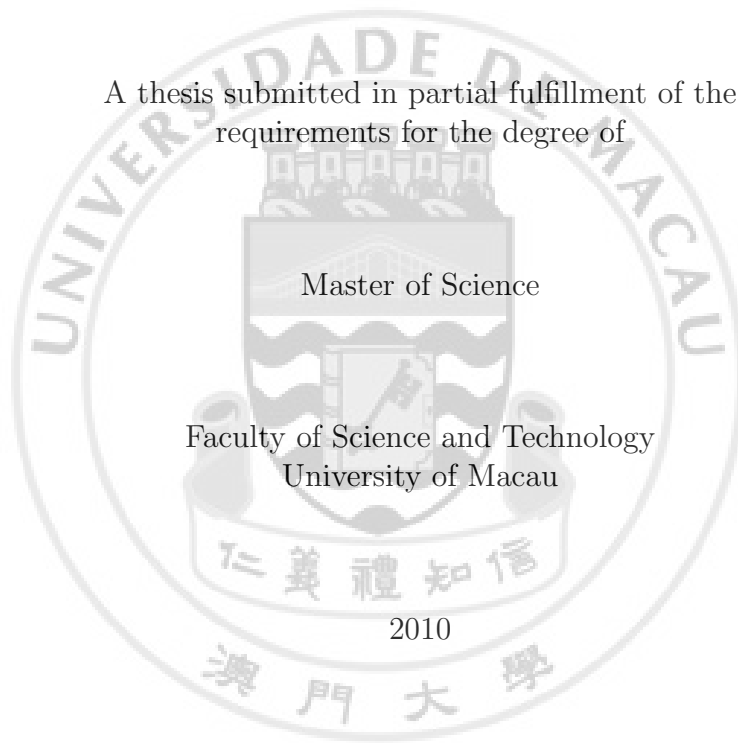
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A thesis submitted in partial fulfillment of the
requirements for the degree of

Master of Science

Faculty of Science and Technology
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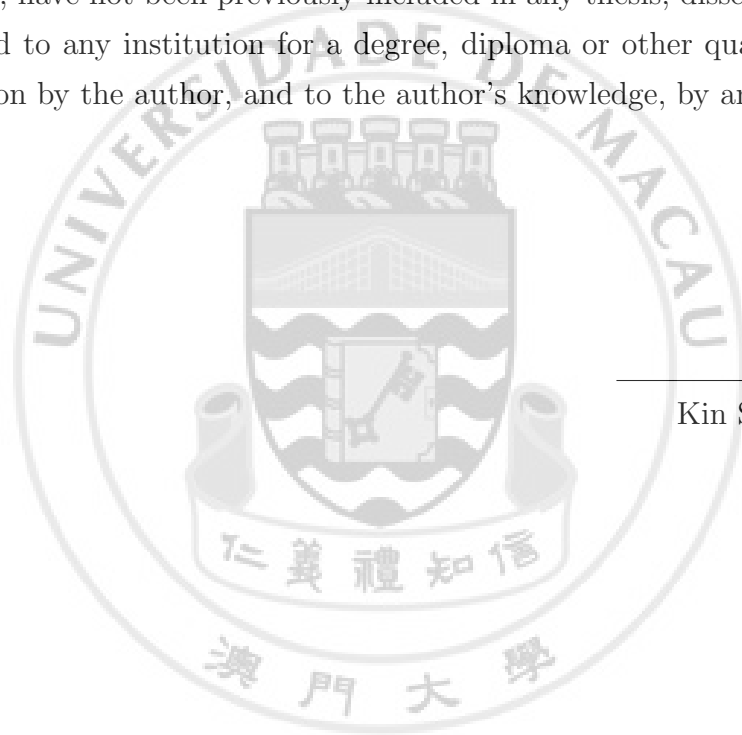


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DECLARATION

The author declares that this thesis represents his own work with Professor Che Man Cheng and Dr. Ieng Tak Leong, the author's supervisor and co-supervisor. All the work is done under the supervision of Professor Cheng and Dr. Leong during the period 2008-2010 for the degree of Master of Science in Mathematics at the University of Macau. The results in this thesis, unless otherwise stated or indicated, have not been previously included in any thesis, dissertation or report submitted to any institution for a degree, diploma or other qualification, or for publication by the author, and to the author's knowledge, by anyone else.



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Abstract

Norm Inequalities for Commutators

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Thesis Supervisors:

Professor Che Man Cheng, Dr. Ieng Tak Leong

Master of Science in Mathematics

It has been conjectured and proved that

$$\|XY - YX\|_F \leq \sqrt{2}\|X\|_F\|Y\|_F$$

for any $n \times n$ complex matrices X and Y , where $\|\cdot\|_F$ denotes the Frobenius norm. A characterization of those pairs of matrices that satisfy the inequality with equality has also been found. Thereafter, Audenaert gave another proof for the inequality by means of what he called the matrix version of variance. Based on his proof, we find another proof for the equality cases in Chapter 2.

Audenaert also showed that

$$\|XY - YX\|_F \leq \sqrt{2}\|X\|_F\|Y\|_{(2),2},$$

where $\|\cdot\|_{(2),2}$ denotes the $(2, 2)$ -norm. In Chapter 3 we characterize the pairs of matrices which satisfy the inequality with equality. Furthermore, we extend this inequality to other Schatten p -norms in Chapter 4.

On the other hand, Böttcher and Wenzel proved that for any unitarily invariant norm $\|\cdot\|$,

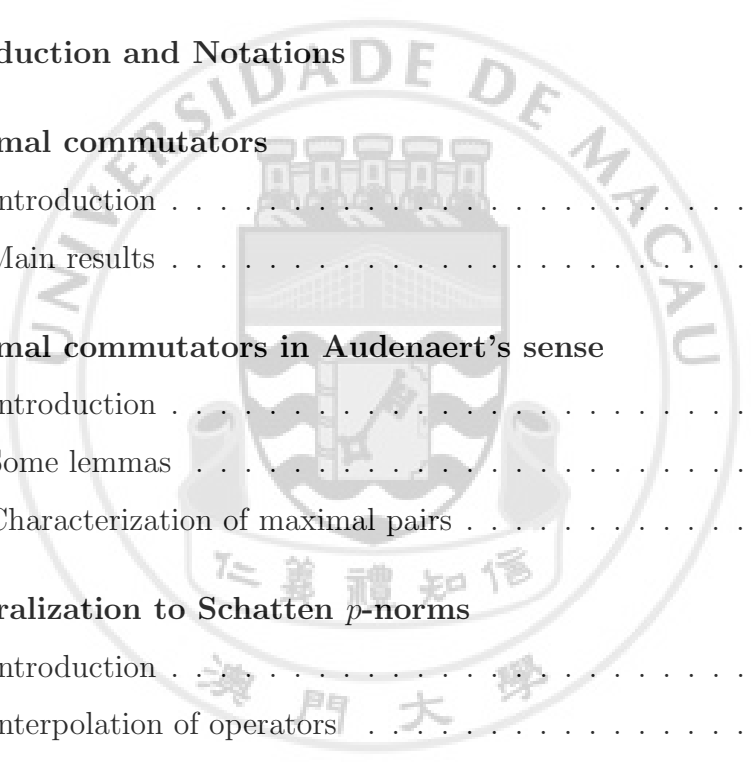
$$\sup \left\{ \frac{\|XY - YX\|}{\|X\|\|Y\|} : X \text{ and } Y \text{ are } n \times n \text{ nonzero complex matrices} \right\} = C \geq \sqrt{2}.$$

They also asked whether the Frobenius norm is the only one having such property.

In Chapter 5 we answer the question by showing that the dual norm of the $(2, 2)$ -norm also has the property that $C = \sqrt{2}$.



Contents



1	Introduction and Notations	1
2	Maximal commutators	5
2.1	Introduction	5
2.2	Main results	5
3	Maximal commutators in Audenaert's sense	12
3.1	Introduction	12
3.2	Some lemmas	13
3.3	Characterization of maximal pairs	20
4	Generalization to Schatten p-norms	28
4.1	Introduction	28
4.2	Interpolation of operators	29
4.3	Better bounds for commutators	33
5	Unitarily invariant norm with minimum bound	35
5.1	Introduction	35
5.2	Properties of unitarily invariant norms	36
5.3	Dual of $(2,2)$ -norm	37
5.4	Equality cases	40
	Bibliography	45

Acknowledgements

I would like to thank Professor Che Man Cheng and Dr. Ieng Tak Leong, my supervisors, for their suggestions and constant support during this research. I am also thankful to Io Kei Lok; he shared with me a lot of his ideas and results, which is crucial to the completion of this research.

Of course, I am grateful to my parents for their encouragement and love.

Finally I wish to thank Professor Xiao Qing Jin, Professor Tao Qian, Professor Deng Ding, Dr. Hai Wei Sun, and Dr. Seak Weng Vong, Chon Ip Chao, Sio Chong U, and Chou Hei Chio for their valuable advice and assistance in my mathematics studies.

