Preconditioning Techniques for a Family of Toeplitz-Like Systems with Financial Applications

by

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Abstract

The field of option pricing has grown very rapidly over the last 30 years. Many academic researchers and financial practitioners in disciplines such as mathematics, finance, economics, business, physics, computer, etc., have been working on this interesting area. In this PhD thesis, the author, from a mathematics discipline, employs modern Krylov subspace methods with preconditioning techniques to solve a family of Toeplitz-like systems from option pricing.

The thesis is divided into three parts. First, we introduce the definition of a family of generating functions (FGF) [119] of Toeplitz matrices, which is a generalization of the generating function of the Toeplitz matrix. The FGF has an important application in pricing derivatives. The pricing of a European call option under Merton's jump-diffusion model, under certain assumptions, can be described by a partial integro-differential equation (PIDE) without a convection term. We then use the FGF to analyze the convergence rate of the preconditioned conjugate gradient (PCG) method with Strang's circulant preconditioner which is useful for solving Toeplitz systems arising from a discretization of the PIDE. We show that if the FGF meets certain conditions, then the spectrum of the preconditioned matrix is clustered around 1, and the smallest eigenvalue of the preconditioned matrix is uniformly bounded away from 0. It follows that the convergence rate of the PCG method is superlinear. The FGF provides a rigorous theoretical basis for the convergence analysis of the conjugate gradient (CG) method. All aspects related to the generating function could be extended to the FGF.

Second, we solve a nonsymmetric Toeplitz system resulting from the finite difference discretization of a PIDE, which is led by a European call option under Merton's jump-diffusion model. In Merton's model, the jump magnitude distribution is normal with mean μ_J and standard deviation σ_J. When $\mu_J = 0$, discretizing the PIDE without the convection term yields a symmetric Toeplitz system [102, 119]. While for $\mu_J \neq 0$, the resulting system is a nonsymmetric Toeplitz system. In [102, 119], the authors only consider the case $\mu_J = 0$. In this part, we discuss a more general case with $\mu_J \neq 0$. The conjugate gradient normal equation residual method [53, 101] is employed. To speed up the numerical process, preconditioning techniques are used. We apply the CG method to a normalized preconditioned system [29]. Strang's circulant preconditioner [111] and a tri-diagonal preconditioner

are studied. By using the definition of the FGF introduced in [119], we show that the spectrum of the normalized preconditioned matrix with Strang's preconditioner is clustered around 1 and the smallest eigenvalue is uniformly bounded away from 0. It follows that the convergence rate of the CG method when applied to such a normalized preconditioned system is superlinear [25]. Similar results are also obtained for the tri-diagonal preconditioner. We see from the numerical results in Section 6.2 that both preconditioners work very well.

Third, we exploit quadratic finite element (FE) with preconditioning for option pricing problems in the Stochastic Volatility with correlated and Contemporaneous Jumps in return and variance (SVCJ) model. The value function for a European vanilla option or barrier option satisfies a PIDE. After spatial discretizations of a variational formulation of the PIDE by quadratic FE, we obtain an ordinary differential equation (ODE) system (which is usually referred to as a semi-discretization of the variational problem). An implicit-explicit Euler based extrapolation scheme proposed by Feng and Linetsky [51] is used to integrate this ODE system in time. For the solutions of the resulting linear systems we adopt the bi-conjugate gradient stabilized (BICGSTAB) method. Since the coefficient matrix of the resulting linear system is block penta-diagonal with penta-diagonal blocks and has certain block structure, a Schur complement preconditioner is designed to accelerate the convergence of the BICGSTAB method. Numerical experiments demonstrate that the combination of quadratic FE, extrapolation scheme, and preconditioning techniques is very efficient to the option pricing problems in the SVCJ model. We expect this combination to be very attractive to financial engineering modelers.

In summary, the major contributions of the thesis are: the introduction of the definition of FGF for Hermitian Toeplitz systems, and the FGF provides a rigorous theoretical basis for the convergence analysis of the CG method; the generalization of the definition of FGF to non-Hermitian Toeplitz systems, and the FGF provides a rigorous theoretical basis for the convergence analysis of the CG method when applied to a normalized preconditioned system, which is led by a European call option under Merton's jump-diffusion model with $\mu_J \neq 0$; the use of the quadratic FE method and the Schur complement preconditioner for European option pricing problems under the SVCJ model.

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The author declares that this thesis represents his own work with Professor Xiao-Qing JIN, the author's supervisor. All the work is done under the supervision of Professor JIN during the period 2007–2010 for the degree of Doctor of Philosophy in Mathematics at the University of Macau. The results in this thesis, unless otherwise stated or indicated, have not been previously included in any other thesis, dissertation, or report submitted to any institution for a degree, diploma, or other qualification, or for publication by the author, and to the author's best knowledge, by anyone else.



Ying-Ying ZHANG

To

My Parents

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Abbreviations

AMS	American Mathematical Society
AMSS	Academy of Mathematics and Systems Sciences
BC	Block Circulant
BDF2	second order Backward Difference Formula
BICG	BI-Conjugate Gradient
BICGSTAB	BI-Conjugate Gradient STABilized
BIT	Nordisk **T**idskrift for **I**nformations **B**ehandling
BSDE	Backward Stochastic Differential Equation
BSOR	Block Successive Over-Relaxation
BT	Block Toeplitz
BTTB	Block Toeplitz with Toeplitz Blocks
CAS	Chinese Academy of Sciences
CB	Circulant Blocks
CBOE	Chicago Board of Options Exchange
CG	Conjugate Gradient
CGMY	Carr, Geman, Madan, and Yor
CGNE	Conjugate Gradient on the Normal Equations
CGNR	Conjugate Gradient on the Normal Residuals
CGS	Conjugate Gradient Squared
CIR	Cox-Ingersoll-Ross
CPU	Central Processing Unit
DAX	Deutscher Aktien indeX
DB	Double Barrier
DBP	Double-Barrier Put
DM	Deutsche Mark
EMM	Equivalent Martingale Measure
EURANDOM	European Institute for Statistics, Probability, Stochastic Operations Research and its Applications
FDM	Freiburg center for Data analysis and Modeling
FE	Finite Element

FFT	Fast Fourier Transform
FGF	Family of Generating Functions
GB	GigaByte
GH	Generalized Hyperbolic
GHz	GigaHertz
GIG	Generalized Inverse Gaussian
GMRES	Generalized Minimal RESidual
GQ	Gaussian Quadrature
IEEE	Institute of Electrical and Electronics Engineers
IG	Inverse Gaussian
i.i.d.	independent and identically distributed
IMA	Institute of Mathematics and its Applications
IMAT	Master program in MAThematics
IMEX	IMplicit-EXplicit
Iter_a	**a**verage number of inner **itera**tions
IVP	Initial Value Problem
MINRES	MINimum RESidual
NIG	Normal Inverse Gaussian
ODE	Ordinary Differential Equation
OTC	Over-The-Counter
PBICGSTAB	Preconditioned BI-Conjugate Gradient STABilized
PC	Personal Computer
PCG	Preconditioned Conjugate Gradient
PDE	Partial Differential Equation
PIDE	Partial Integro-Differential Equation
PPT	PowerPoinT
QMR	Quasi-Minimal Residual
RAM	Random Access Memory
RIUPEEEC	Regional Inter-University Postgraduate Electrical and Electronics Engineering Conference

SDE	Stochastic Differential Equation
SIAM	Society for Industrial and Applied Mathematics
SOR	Successive Over-Relaxation
SPD	Symmetric Positive Definite
SSOR	Symmetric SOR
SV	Stochastic Volatility
SVCJ	Stochastic Volatility with correlated and Contemporaneous Jumps in return and variance
SVJ	Stochastic Volatility with Jumps in return
SYMMLQ	SYMMetric LQ
TM	TradeMark
VG	Variance Gamma

Symbols, notations, and definitions

We use the following symbols, notations, and definitions throughout the thesis [25].

- Let $\mathbb{R}$ be the set of real numbers; $\mathbb{C}$, the set of complex numbers; and $\mathbf{i} \equiv \sqrt{-1}$. The symbol $\mathbb{R}^n$ denotes the set of real n-vectors and $\mathbb{C}^n$, the set of complex n-vectors. Vectors are almost always column vectors. We use $\mathbb{R}^{m\times n}$ to denote the linear vector space of all $m \times n$ real matrices and $\mathbb{C}^{m\times n}$, the linear vector space of all $m\times n$ complex matrices. The symbol $\dim(S)$ denotes the dimension of a linear vector space S.

- The upper case letters A, B, C, Δ, Λ, etc., denote matrices and the boldface lower case letters $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, etc., denote vectors.

- Let $[A]_{ij} = a_{ij}$ denote the (i,j)th entry in a matrix $A = [a_{ij}]$. Let A^T be the transpose of A; A^*, the conjugate transpose of A; $\operatorname{rank}(A)$, the rank of A; and $\operatorname{tr}(A) \equiv \sum_{i=1}^{n} a_{ii}$, the trace of A. An $n \times n$ diagonal matrix is denoted by

$$\operatorname{diag}(a_{11}, a_{22}, \ldots, a_{nn}) = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & a_{nn} \end{bmatrix}.$$

  We use notation I_n for an $n \times n$ identity matrix. When there is no ambiguity, we shall write it as I. The symbol $\mathbf{e}_j$ denotes the jth unit vector, i.e., the jth column vector of I.

- A matrix $A \in \mathbb{R}^{n\times n}$ is symmetric if $A^T = A$. A symmetric matrix A is positive definite (semidefinite) if $\mathbf{x}^T A\mathbf{x} > 0$ (≥ 0) for any nonzero vector $\mathbf{x} \in \mathbb{R}^n$. A matrix $A \in \mathbb{C}^{n\times n}$ is Hermitian if $A^* = A$. A Hermitian matrix A is positive definite (semidefinite) if $\mathbf{x}^* A\mathbf{x} > 0$ (≥ 0) for any nonzero vector $\mathbf{x} \in \mathbb{C}^n$.

- A number $\lambda \in \mathbb{C}$ is an eigenvalue of $A \in \mathbb{C}^{n\times n}$ if there exists a nonzero vector $\mathbf{x} \in \mathbb{C}^n$ such that $A\mathbf{x} = \lambda\mathbf{x}$, where $\mathbf{x}$ is called the eigenvector of A associated with λ. It is well known that the eigenvalues of all Hermitian matrices are real. Let $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ denote the smallest and largest eigenvalues of a Hermitian matrix A respectively. We use $\rho(A) \equiv \max|\lambda_i(A)|$ to denote the spectral radius of A where $\lambda_i(A)$ goes through the spectrum of A. Recall that the spectrum of A is the set of all the eigenvalues of A.

- We use A_{mn} to denote any $m \times m$ block matrix with $n \times n$ blocks and $[A_{mn}]_{i,j;k,l}$ to denote the (i,j)th entry in the (k,l)th block of A_{mn}.

- We use $\|\cdot\|$ to denote a norm of vector or matrix. The symbols $\|\cdot\|_1$, $\|\cdot\|_2$, $\|\cdot\|_\infty$ denote the p-norm with $p = 1, 2, \infty$, respectively.

– Let $\mathbf{C}_{2\pi}$ be the space of all 2π-periodic continuous real-valued functions $f(x)$ and $\mathbf{C}^{+}_{2\pi}$, the subset of all functions $f(x) \geq 0$ in $\mathbf{C}_{2\pi}$ which are not identically zero. We use $\mathbf{C}_{2\pi\times 2\pi}$ to denote the space of all 2π-periodic (in each direction) continuous real-valued functions $f(x, y)$.