High Order Compact Scheme and Its Applications in Computational Finance

by

Spike Tsz-Ho Lee

Master of Science in Mathematics

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DECLARATION

The author declares that this thesis represents his own work with Dr. Hai-Wei Sun, the author's supervisor. All the work is done under the supervision of Dr. Sun during the period 2007-2010 for the degree of Master of Science in Mathematics at the University of Macau. The results in this thesis, unless otherwise stated or indicated, have not been previously included in any thesis, dissertation or report submitted to any institution for a degree, diploma or other qualification, or for publication by the author, and to the author's knowledge, by anyone else.



Spike Tsz-Ho Lee

To

My Dearly Beloved Family

High order compact scheme and its applications in computational finance

by Spike Tsz-Ho Lee

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Introduction

In this thesis, high order compact (HOC) schemes are considered in the option pricing problem from computational finance. In particular, the fourth order compact (FOC) scheme is used to discretize the partial integro-differential equation (PIDE) arising from the option pricing framework. The FOC scheme is well known for its high order accuracy, ability to suppress oscillations, and adaptability with local mesh refinement strategy. These properties will show their competence for pricing options. Various temporal integrators are proposed to couple with the FOC scheme. Numerical experiments show the superb performance of the these combined methods. The thesis is divided into four chapters. A brief introduction to option pricing and some related tools are given in chapter 1.

In chapter 2, the local mesh refinement strategy is implemented in space in order to subdue the non-smoothness of the payoff function. Combined with the extrapolated implicit-explicit (IMEX) scheme, the FOC scheme with local mesh refinement strategy is shown to attain fourth order accuracy in numerical experiments.

In chapter 3, the boundary value method (BVM) is proposed to join forces with the FOC scheme. The non-smoothness in the payoff function also disturbs the time direction, therefore two kinds of startup procedure are proposed to smoothen the oscillations in the initial time layers. Two efficient preconditioners are used to solve the resulting large-scale linear systems.

In chapter 4, the exponential integration scheme (ETI) is used to integrate the FOC-discretized ODE system. By the ETI scheme, the time direction of PIDE is directly tackled by a "one step" formula, i.e., discretization in time is not needed. Technically speaking, the ETI scheme involves the evaluation of Toeplitz matrix exponential (TME), which is the exponential of a Toeplitz matrix multiplied by a vector. Therefore the approximation to TME is first studied, then the ETI scheme is applied to solve the option pricing problem.

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