

**On some norm inequalities involving the
commutator and $XY - YX^T$**

by

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Master of Science in Mathematics

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**Faculty of Science and Technology
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Abstract

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Let X and Y be any $n \times n$ complex matrices, $\|\cdot\|_F$ be the Frobenius norm and $\|\cdot\|_{(2),2}$ be the $(2,2)$ -norm defined by $\|X\|_{(2),2} = (s_1^2(X) + s_2^2(X))^{\frac{1}{2}}$, where $s_1(X) \geq \cdots \geq s_n(X)$ are the singular values of X . Based on the paper [A. Böttcher and D. Wenzel, The Frobenius norm and the commutator, Linear Algebra Appl. 429 (2008) 1864-1885], a common approach is found for proving the following three inequalities:

$$\|XY - YX\|_F \leq \sqrt{2}\|X\|_{(2),2}\|Y\|_F,$$

$$\|XY - YX^T\|_F \leq \sqrt{2}\|X\|_{(2),2}\|Y\|_F$$

and

$$\|XY - YX^T\|_F \leq \sqrt{2}\|X\|_F\|Y\|_{(2),2}.$$

The first two are known and the third one is new. Their equality cases are also determined. In particular it is found that, while there are common analogous equality cases, the third inequality admits an equality case that the other two do not have as a corresponding counterpart.

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