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Abstract

Equality cases of some Hölder-type norm inequalities

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In this thesis, we study the equality cases of some Hölder-type norm inequalities in matrix spaces.

Let M_n denote the space of $n \times n$ complex matrices. A norm $\| \ \|$ on M_n is said to be unitarily invariant if

$$\|UAV\| = \|A\|$$

for all $A, U, V \in M_n$ with U and V unitary. Let $A = (a_{ij})$ and $B = (b_{ij})$ be matrices in M_n. As usual, $A \circ B = (a_{ij} b_{ij})$ denotes the Hadamard (entrywise) product of A and B. Define $|A| \equiv (A^* A)^{\frac{1}{2}}$, where A^* is the conjugate transpose of A.

Let $\| \ \|$ be a unitarily invariant norm on M_n and let p, q and r be positive numbers with $\frac{1}{p} + \frac{1}{q} = 1$. The following Hölder-type norm inequalities are known:

$$\left\| |X|^r \right\| \le \left\| L^{\frac{rp}{2}} \right\|^{\frac{1}{p}} \left\| M^{\frac{rq}{2}} \right\|^{\frac{1}{q}},$$

where $\left(\begin{array}{c|c} L & X \\ \hline X^* & M \end{array}\right)$ is a positive semidefinite matrix where $L, M, X \in M_n$;

$$\left\| |AB|^r \right\| \le \left\| |A|^{pr} \right\|^{\frac{1}{p}} \left\| |B|^{qr} \right\|^{\frac{1}{q}}, \quad A, B \in M_n;$$

$$\left\| |AXB|^r \right\| \le \left\| |A^p X|^r \right\|^{\frac{1}{p}} \left\| |XB^q|^r \right\|^{\frac{1}{q}},$$

where A and B are positive semidefinite and $A, B, X \in M_n$; and

$$\left\| |A \circ B|^r \right\| \le \left\| |A|^{pr} \right\|^{\frac{1}{p}} \left\| |B|^{qr} \right\|^{\frac{1}{q}}, \quad A, B \in M_n, \quad r \ge 1.$$

Though the inequalities are known, their equality cases are not given in literature and do not follow directly from the proofs. In this thesis, we characterize their equality cases.