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Abstract

Equality Cases for Some Inequalities Involving the Hadamard  
Product of Hermitian Matrices

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In this thesis, we study the equality cases for some inequalities involving the Hadamard product of Hermitian matrices.

Let  $\mathbf{C}_{n \times n}$  denote the set of all  $n \times n$  complex matrices, and  $\mathbf{H}_n$  be the set of all  $n \times n$  Hermitian matrices. The Hadamard (entrywise) product of  $A = (a_{ij})$ ,  $B = (b_{ij}) \in \mathbf{C}_{n \times n}$  is defined and denoted by

$$A \circ B = (a_{ij}b_{ij}) \in \mathbf{C}_{n \times n}.$$

For any  $A \in \mathbf{H}_n$  (resp.  $A \in \mathbf{C}_{n \times n}$ ), let  $\lambda_1(A) \geq \dots \geq \lambda_n(A)$  (resp.  $\sigma_1(A) \geq \dots \geq \sigma_n(A)$ ) denote the eigenvalues (resp. singular values) of  $A$  and let  $\lambda(A) = (\lambda_1(A), \dots, \lambda_n(A))^t$  (resp.  $\sigma(A) = (\sigma_1(A), \dots, \sigma_n(A))^t$ ).

For any  $n \times n$  positive definite matrices  $A$  and  $B$ , it is known that the following inequalities in multiplicative form hold :

$$\prod_{i=k}^n \lambda_i(A \circ B) \geq \prod_{i=k}^n \lambda_i(AB), 1 \leq k \leq n,$$

$$\prod_{i=k}^n \lambda_i(A \circ B) \geq \prod_{i=k}^n \lambda_i(AB^t), 1 \leq k \leq n.$$

In Chapter 2, we characterize the equality cases of the above inequalities.

For general matrices  $A, B \in \mathbf{C}_{n \times n}$ , it is known that

$$\sum_{i=1}^k \sigma_i(A \circ B) \leq \sum_{i=1}^k \sigma_i(A) \sigma_i(B), \quad 1 \leq k \leq n.$$

Moreover, the equality cases are also known. However, when  $A$  and  $B$  are restricted to be Hermitian, further consideration is needed. In Chapter 3, we continue the study and characterize the equality of the above inequalities for Hermitian matrices.

A brief introduction of the inequalities is given in Chapter 1.