

Introduction

In the complex plane, Schwarz kernel is the sum of the Poisson kernel and the so-called conjugate Poisson kernel. Given a real function u , harmonic in the unit disk and continuous on the closure of unit disk, we may obtain through Schwarz kernel an analytic function f in the unit disk whose real part coincides with u . Moreover, the nontangential boundary function of conjugate Poisson integral of u coincides with the circular Hilbert transform of u . See [9]. This thesis is devoted to studying Schwarz kernels on unit spheres in $\mathbf{R}^n$ with Clifford setting. Among several candidates, we finally define the Schwarz kernels on unit spheres which is the sum of Poisson kernel for unit ball and a particular harmonic conjugate of Poisson kernel. Moreover, such defined Schwarz kernels have the similar properties as those in the complex plane we mentioned above.

In Chapter 1, we discuss methods of deriving Poisson kernel, the real part of Schwarz kernel. As is well-known, Poisson kernel is fundamental in both harmonic analysis and Fourier analysis. The solution of Dirichlet problem is the Poisson integral of the given boundary function, both in the complex plane and higher dimensional spaces. In the complex plane, there are several ways to produce the Poisson kernel, including one from the Abelian sum of Fourier series (see [17]), one from mean value property for harmonic function(see [9]), and one from the Poisson kernel for upper half plane (see [9]). The main work of this chapter is the adaptation of the dimension 2 case to higher dimensional spaces ($\mathbf{R}_1^n$ or $\mathbf{R}^n$) in obtaining the corresponding Poisson kernel. For completeness, we also include the result of [8] which provides us the second way to produce Poisson kernel in $\mathbf{R}^n$. Systematically discussing the deductions of the Poisson kernel helps us to understand more about the Schwarz kernel.

In Chapter 2, our main task is to find Schwarz kernel through defining the so-called spherical Hilbert transform. There are a number of approaches of the concept of harmonic conjugate functions in higher dimensional spaces. We call the first type vector harmonic conjugate functions. The conjugate Poisson kernel for upper half spaces is of this kind. Cartesian harmonic conjugate functions, the second type, provides a viewpoint on defining the harmonic conjugate for a $\mathbf{R}^{(n-1)}$ valued function. See [12],[13]. In [13], there is another kind of harmonic conjugate for a real-valued function called angular harmonic conjugate functions which is the spherical approach of Cartesian harmonic conjugate functions. We also provide some formulas on attaining these three types of harmonic conjugate functions for a given $\mathbf{R}^{(n-1)}$ -valued harmonic function or real-valued harmonic function in some particular domains, among which [13] and [15] together give us the clear expression of the angular harmonic conjugate of Poisson kernel. As main result of this chapter, we define the spherical Hilbert transform in the form of conjugate Fourier series where the associated kernel is the angular harmonic conjugate function of Poisson kernel, called conjugate Poisson kernel. Moreover, the spherical Hilbert transform of an L^2 function is the L^2 boundary of the conjugate Poisson integral of it. In this context, the Schwarz kernel on the unit sphere $\mathbf{S}^{n-1}$ takes the form,

$$S(\underline{y}, \underline{x}) = \frac{1}{\Omega_{n-1}} \frac{1 - |\underline{x}|^2}{|\underline{y} - \underline{x}|^n} + \frac{n}{\Omega_{n-1}} \int_0^1 \frac{1 - r^2 s^2}{|s\underline{x} - \underline{y}|^{n+2}} s^{n-2} ds \underline{x} \wedge \underline{y}.$$

By calculating the Abelian sum of conjugate Fourier series, an alternative form of Schwarz kernel is attained as follows,

$$S(\underline{y}, \underline{x}) = \frac{1}{\Omega_{n-1}} \frac{1 - |\underline{x}|^2}{|\underline{y} - \underline{x}|^n} + \frac{1}{\Omega_{n-1}} \left(\frac{2}{|\underline{y} - \underline{x}|^n} - \frac{n-2}{r^{n-1}} \int_0^r \frac{\rho^{n-2}}{|\underline{y} - \rho\omega|^n} d\rho \right) \underline{x} \wedge \underline{y},$$

where $x = r\omega$, $\omega \in \mathbf{S}^{n-1}$.